FP1 Jan 09

1) $2x^3-8x^2+7x-3=(x-3)(2x^2+Ax+1)$

$Ax^2-6x^2 \equiv -8x^2 \Rightarrow A=-2$

$\Rightarrow (x-3)(2x^2-2x+1)$

2) $\sum 6r^2+4r-1 = 6\sum r^2+4\sum r - \sum 1$

$=6\left(\frac{1}{6}n(n+1)(2n+1)\right)+4\left(\frac{1}{2}n(n+1)\right)-n$

$=n\left[(n+1)(2n+1)+2n+2-1\right]$

$=n\left[2n^2+5n+2\right] = n(2n+1)(n+2)$ #

b) $\sum_{11}^{20} 6r^2+4r-1 = 20(41)(22)-10(21)(12)$

$=\underline{15520}$

3) $y = \frac{5}{t} = \frac{25}{5t} \Rightarrow y = \frac{25}{x} \Rightarrow xy = 25$

b) $A(5,5)$ $B(25,1)$ Midpoint $AB = (15,3)$

4) $n=1$ $\sum_{1}^{1} \frac{1}{r(r+1)} = \frac{1}{1(2)} = \frac{1}{2}$ $\frac{n}{n+1} = \frac{1}{1+1} = \frac{1}{2}$ tr

$n=k$ $\sum_{1}^{k} \frac{1}{r(r+1)} = \frac{k}{k+1}$

$n=k+1$ $\sum_{1}^{k+1} \frac{1}{r(r+1)} = \frac{k+1}{k+2}$

$$\sum_{1}^{k+1} \frac{1}{r(r+1)} = \sum_{1}^{k} \frac{1}{r(r+1)} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)}$$

$$= \frac{\cancel{(k+1)}(k+1)}{\cancel{(k+1)}(k+2)} = \frac{k+1}{k+2}$$ as required.

true for $n=1$, true for $n=k$ if true for $n=k+1$
$\therefore$ by induction true for all $n \in \mathbb{Z}^+$.

5) $f(x) = 3\sqrt{x} + \frac{18}{\sqrt{x}} - 20$ $f(1.1) = 0.31 > 0$
$f(1.2) = -0.28 < 0$

Sign change $\Rightarrow \alpha \in [1.1, 1.2]$

b) $f(x) = 3x^{\frac{1}{2}} + 18x^{-\frac{1}{2}} - 20 \Rightarrow f'(x) = \frac{3}{2}x^{-\frac{1}{2}} - 9x^{-\frac{3}{2}}$

c) $x_1 = 1.1 - \frac{f(1.1)}{f'(1.1)} = \underline{1.15}$

6) $u_1 = 6$ $u_{n+1} = 6u_n - 5$ $u_n = 5(6^{n-1}) + 1$

$u_1 = 6$ $u_1 = 5(6^{1-1}) + 1 = 5 + 1 = 6$
$u_2 = 6(6) - 5 = 31$ $u_2 = 5(6^{2-1}) + 1 = 30 + 1 = 31$

$n = k$ $u_k = 5(6^{k-1}) + 1$

$n = k+1$ $u_{k+1} = 6u_k - 5 = 6(5(6^{k-1}) + 1) - 5$
$= 5(6 \times 6^{k-1}) + 6 - 5$
$= 5(6^k) + 1$

$u_{k+1} = 5(6^{k+1-1}) + 1 = 5(6^k) + 1$ as required.

true for $n=1$, true for $n=2$, true for $n=k+1$ if true for $n=k$
$\therefore$ by Induction true for all $n \in \mathbb{Z}^+$

7) $X = \begin{pmatrix} 2 & a \\ -1 & -1 \end{pmatrix}$ $\det X = -2 + a \Rightarrow X^{-1} = \frac{1}{a-2}\begin{pmatrix} -1 & -a \\ 1 & 2 \end{pmatrix}$

b) $\frac{1}{a-2}\begin{pmatrix} -1 & -a \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 2 & a \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \frac{-1}{a-2} + 2 = 1$

$\frac{-1}{a-2} = -1 \Rightarrow a - 2 = 1$
$\underline{a = 3}$

8) $y^2 = 4ax$ $\quad \frac{d}{dx}y^2 = \frac{d}{dx}(4ax) \Rightarrow 2y\frac{dy}{dx} = 4a$

$\Rightarrow \frac{dy}{dx} = \frac{2a}{y} \Rightarrow m_t = \frac{2a}{2aq} = \frac{1}{q}$

$y - 2aq = \frac{1}{q}(x - aq^2)$

$\times q$ $\quad qy - 2aq^2 = x - aq^2 \Rightarrow yq = x + aq^2$ #

b) meets y when $x=0 \Rightarrow y\cancel{q} = aq^{\cancel{2}} \Rightarrow y = aq$

$R(0, aq)$ perp $\Rightarrow m = -q \quad y - aq = -q(x - 0)$

$\Rightarrow y = -qx + aq$.

c) focus $(a, 0)$ $\quad x = a \quad y = -qa + aq = 0$ #

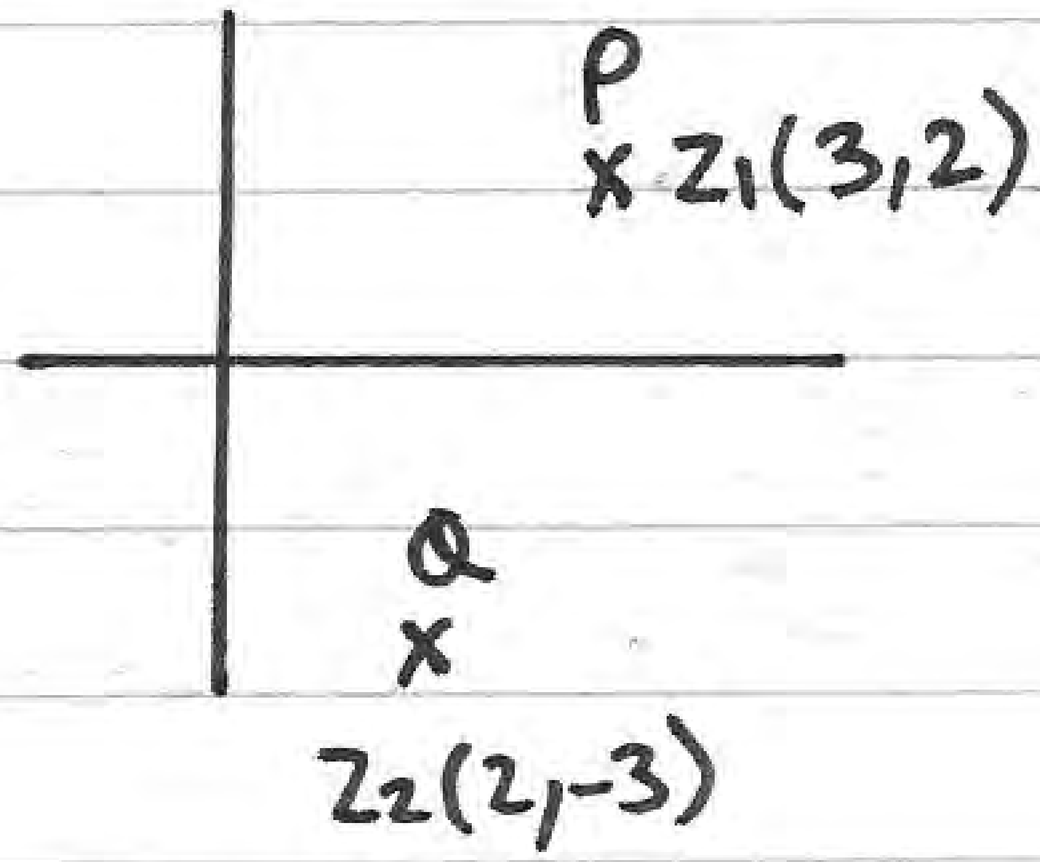
d) at directrix $x = -a \Rightarrow y = -q(-a) + aq = aq + aq$

$\Rightarrow y = 2aq$

$(-a, 2aq)$

9) $Z_2 = \frac{12-5i}{3+2i} \frac{\times(3-2i)}{\times(3-2i)} = \frac{36-24i-15i+10i^2}{9-4i^2} = \frac{26-39i}{13}$

$Z_2 = 2-3i$

b) 

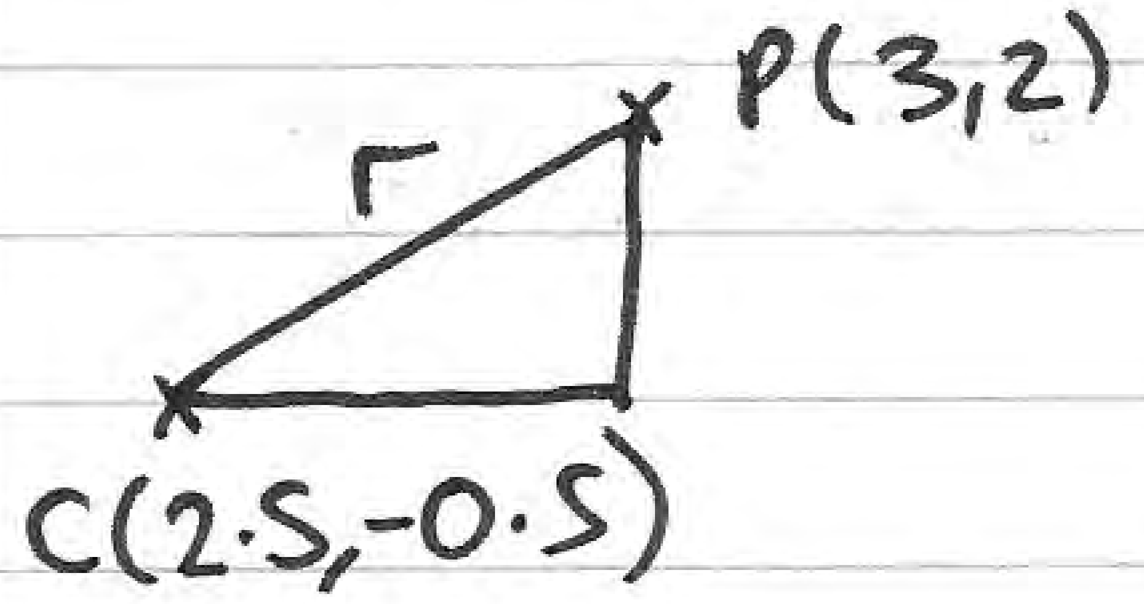
$m_{OP} = \frac{2}{3}$ $\quad m_{OQ} = \frac{-3}{2}$

$m_{OP} \times m_{OQ} = \frac{2}{3} \times -\frac{3}{2} = -1$

hence perp $\Rightarrow \angle POQ = \frac{\pi}{2}$

c) $\angle POQ = \frac{\pi}{2} \Rightarrow PQ$ must be the diameter

$\Rightarrow$ midpoint PQ = centre $= \frac{5}{2} - \frac{1}{2}i$



$r^2 = \left(\frac{1}{2}\right)^2 + \left(\frac{5}{2}\right)^2 \Rightarrow r = \sqrt{\frac{1}{4} + \frac{25}{4}}$

$\Rightarrow r = \frac{1}{2}\sqrt{26}$

10) $A = \begin{pmatrix} 3\sqrt{2} & 0 \\ 0 & 3\sqrt{2} \end{pmatrix}$ enlargement, centre origin, scale factor $3\sqrt{2}$

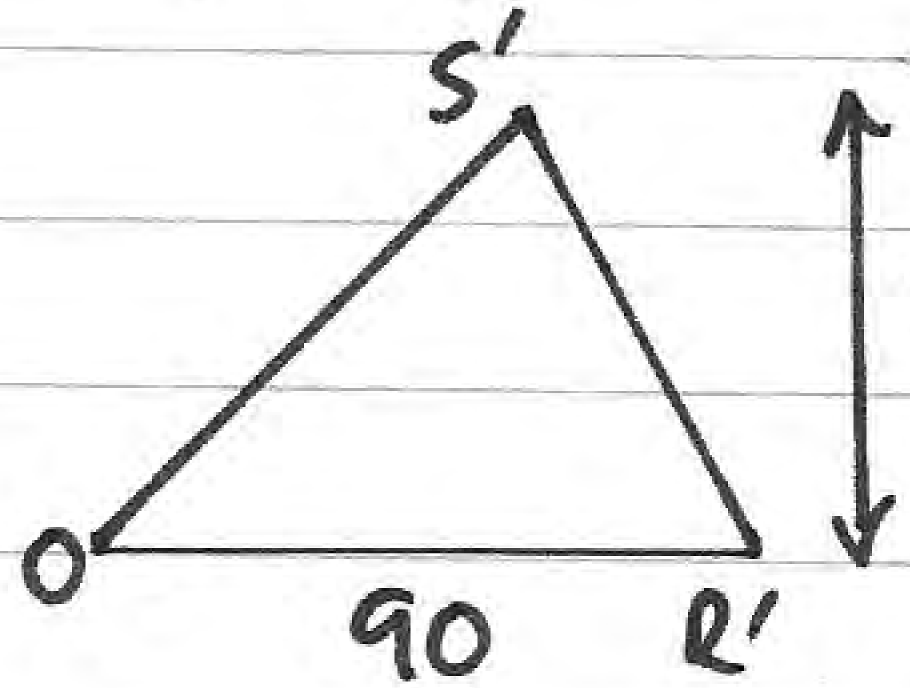
$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ reflection through $y=x$

$C = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$ rotation $45°$, anticlockwise, about origin.

b) $D = CA = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 3\sqrt{2} & 0 \\ 0 & 3\sqrt{2} \end{pmatrix} = \begin{pmatrix} 3 & -3 \\ 3 & 3 \end{pmatrix}$

c) $E = DB = \begin{pmatrix} 3 & -3 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -3 & 3 \\ 3 & 3 \end{pmatrix}$

d) $\begin{pmatrix} -3 & 3 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} 0 & -15 & 4 \\ 0 & 15 & 21 \end{pmatrix} = \begin{pmatrix} 0 & 90 & 51 \\ 0 & 0 & 75 \end{pmatrix}$ $R'(90,0)$ $S'(51,75)$

e) 

$\text{Area} = \frac{6750}{2} = 3375$

$\det E = -9-9 = -18 \Rightarrow \text{Area } ORS = \underline{187.5}$

$\div 18$